



TITLE:

TENTATIVE STUDY ON EQUIVARIANT  
SURGERY OBSTRUCTIONS : FIXED POINT  
SETS OF SMOOTH  $A_5$ -ACTIONS (The  
Topology and the Algebraic Structures of  
Transformation Groups)

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CITATION:

森本, 雅治. TENTATIVE STUDY ON EQUIVARIANT SURGERY OBSTRUCTIONS : FIXED POINT SETS OF SMOOTH  $A_5$ -ACTIONS (The Topology and the Algebraic Structures of Transformation Groups). 数理解析研究所講究録 2014, 1922: 154-165; KJ00009564259.

ISSUE DATE:

2014-10

URL:

<http://hdl.handle.net/2433/223417>

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# TENTATIVE STUDY ON EQUIVARIANT SURGERY OBSTRUCTIONS: FIXED POINT SETS OF SMOOTH $A_5$ -ACTIONS

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**Abstract.** Let  $G$  be the alternating group on 5 letters and let  $F$  be a closed smooth manifold diffeomorphic to the fixed point set of a smooth  $G$ -action on a disk. Marek Kaluba proved that if  $F$  is even dimensional then there exists a smooth  $G$ -action on a closed manifold  $X$  being homotopy equivalent to a complex projective space such that the fixed point set of the  $G$ -action is diffeomorphic to  $F$ . In this paper we discuss whether series of manifolds diffeomorphic or homotopy equivalent to complex projective spaces, real projective spaces, or lens spaces, admit smooth  $G$ -actions with fixed point set diffeomorphic to  $F$ .

## 1. INTRODUCTION

Let  $G$  be a finite group throughout this paper. For a smooth manifold  $M$ , let  $\mathfrak{F}_G(M)$  denote the family of all manifolds  $F$  such that  $F = M^G$  for some smooth  $G$ -action on  $M$ . For a family  $\mathfrak{M}$  of smooth manifolds, let  $\mathcal{F}_G(\mathfrak{M})$  denote the union of  $\mathfrak{F}_G(M)$  with  $M \in \mathfrak{M}$ . Let  $\mathfrak{D}$ ,  $\mathfrak{S}$ , and  $\mathfrak{P}_{\mathbb{C}}$  denote the families of disks, spheres, and complex projective spaces, respectively. B. Oliver [19] completely determined the family  $\mathfrak{F}_G(\mathfrak{D})$  for  $G$  not of prime power order. K. Pawałowski and the author [18, 14] studied  $\mathfrak{F}_G(\mathfrak{S})$  for various Oliver groups  $G$ .

In order to quote a part of Oliver's result on  $\mathfrak{F}_G(\mathfrak{D})$ , we adopt the notation  $\mathcal{G}_{\mathbb{R}}$ ,  $\mathcal{G}_{\mathbb{C}}^c$ ,  $\mathcal{G}_{\mathbb{C}}$  and  $\mathcal{E}$  for the families of all finite groups satisfying the following properties, respectively.

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*2010 Mathematics Subject Classification.* Primary 57S17; Secondary 20C15.

*Key words and phrases.* equivariant manifold, equivariant framed map, fixed point set.

This research was partially supported by JSPS KAKENHI Grant Number 26400090.

- $G \in \mathcal{G}_{\mathbb{R}}$ :  $G$  possesses a subquotient  $K/H$  isomorphic to a dihedral group of order  $2pq$  for some distinct primes  $p$  and  $q$ , where  $H \triangleleft K \leq G$ .
- $G \in \mathcal{G}_{\mathbb{C}}^g$ :  $G$  contains an element  $g$  being conjugate to its inverse of order  $pq$  for some distinct primes  $p$  and  $q$ .
- $G \in \mathcal{G}_{\mathbb{C}}$ :  $G$  contains an element  $g$  of order  $pq$  for some distinct primes  $p$  and  $q$ .
- $G \in \mathcal{E}$ : A Sylow 2-subgroup of  $G$  is not normal in  $G$ , and any element of  $G$  is of prime power order.

Note that  $\mathcal{G}_{\mathbb{R}} \subset \mathcal{G}_{\mathbb{C}}^g \subset \mathcal{G}_{\mathbb{C}}$ . Let  $A_5$  denote the alternating group on 5 letters. Then  $A_5$  belongs to  $\mathcal{E}$ . B. Oliver [19] says that for  $G \in \mathcal{F}_{\mathbb{C}} \cup \mathcal{E}$ , a closed manifold  $F$  belongs to  $\mathfrak{F}_G(\mathcal{D})$  if and only if  $\chi(F) \equiv 1 \pmod{n_G}$  and

$$(1.1) \quad \begin{aligned} & \bullet G \in \mathcal{G}_{\mathbb{R}} \Rightarrow \text{no restrictions on } T(F), \\ & \bullet G \in \mathcal{G}_{\mathbb{C}}^g \setminus \mathcal{G}_{\mathbb{R}} \Rightarrow c_{\mathbb{R}}([T(F)]) \in c_{\mathbb{H}}\left(\widetilde{KS}p(F)\right) + \text{Tor}\left(\widetilde{KU}(F)\right), \\ & \bullet G \in \mathcal{G}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{C}}^g \Rightarrow [T(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \text{Tor}\left(\widetilde{KO}(F)\right), \\ & \bullet G \in \mathcal{E} \Rightarrow [T(F)] \in \text{Tor}\left(\widetilde{KO}(F)\right). \end{aligned}$$

If  $G \in \mathcal{E}$  and  $F \in \mathfrak{F}_G(\mathcal{D})$  then each connected component of  $F$  has same dimension. The Oliver number  $n_G$  above is equal to 1 whenever  $G$  is nonsolvable.

Marek Kaluba [5] obtained the next two theorems concerned with  $\mathfrak{F}_G(\mathfrak{P}_{\mathbb{C}})$ .

**Theorem.** [5, Theorem 2.6] *Let  $G$  be a nontrivial perfect group in the class  $\mathcal{G}_{\mathbb{C}}$  and let  $F$  be a closed manifold in  $\mathfrak{F}_G(\mathcal{D})$ . In the case  $G \in \mathcal{G}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{R}}$ , suppose that some connected component of  $F$  is even dimensional. Then  $F$  belongs to  $\mathfrak{F}_G(\mathfrak{P}_{\mathbb{C}})$ .*

**Theorem.** [5, Theorem 4.11] *Let  $G$  be  $A_5$  and  $F$  a closed manifold in  $\mathfrak{F}_G(\mathcal{D})$ . Suppose that  $F$  is even dimensional. Then  $F$  is diffeomorphic to the fixed point set of a smooth  $G$ -action on a closed manifold  $X$  which is homotopy equivalent to some complex projective space.*

Let  $P_{\mathbb{C}}^k$  (resp.  $P_{\mathbb{R}}^k$ ) denote the complex (resp. real) projective space of complex (resp. real) dimension  $k$ , and let  $\Gamma$  be a cyclic subgroup of  $\mathbb{C}^{\times}$  of order  $\geq 3$ . The orbit space  $L^{2k+1} = S(\mathbb{C}^{k+1})/\Gamma$  is a lens space of dimension  $2k+1$ . Let  $\mathfrak{L}$  be the

family of lens spaces  $L^{2k+1}$ ,  $k = 2, 3, 4, \dots$ . By examining and improving the proof of [5, Theorem 4.11] by M. Kaluba, we obtain the next result.

**Theorem 1.1.** *Let  $G$  be  $A_5$  and  $F$  a closed manifold in  $\mathfrak{F}_G(\mathfrak{D})$ . Then there exists an integer  $N > 0$  possessing the property that for any  $k \geq N$ ,*

- (1)  $F \in \mathfrak{F}_G(D^k)$ ,
- (2)  $F \in \mathfrak{F}_G(S^k)$ ,
- (3) if  $\dim F \equiv 0 \pmod{2}$  then  $F \in \mathfrak{F}_G(P_{\mathbb{C}}^k)$ ,
- (4)  $F \in \mathfrak{F}_G(X_k)$  such that  $X_k$  is a smooth closed manifold homotopy equivalent to  $P_{\mathbb{R}}^k$ ,
- (5) if  $\dim F \equiv 1 \pmod{2}$  then  $F \in \mathfrak{F}_G(Y_k)$  such that  $Y_k$  is a smooth closed manifold homotopy equivalent to  $L^{2k+1}$ .

This result follows from Theorem 3.4. In Theorem 1.1, one may conjecture that  $P_{\mathbb{R}}^k$  and  $L^{2k+1}$  can be chosen as  $X_k$  and  $Y_k$  respectively, but the author cannot prove the conjecture so far.

**Acknowledgment.** The author would like to express his gratitude to Marek Kaluba and Krzysztof Pawałowski for their information related to this research.

## 2. DIMENSION CONDITIONS OF FIXED POINT SETS

Let  $G$  be a finite group. Let  $U$  be a  $G$ -manifold and  $(H, K)$  a pair of subgroups  $H < K \leq G$ . We say that  $U$  satisfies the *gap condition*, *cobordism gap condition*, or *strong gap condition* for  $(H, K)$  if the inequality

$$(2.1) \quad 2 \dim(U^H_i)^K < \dim U^H_i,$$

$$(2.2) \quad 2 (\dim\{(U^H_i)^K \setminus (U^H_i)^{N_G(H)}\} + 1) \leq \dim U^H_i,$$

or

$$(2.3) \quad 2\{\dim(U^H_i)^K + 1\} < \dim U^H_i,$$

holds, respectively, for any connected component  $U^H_i$  of  $U^H$ .

**Proposition 2.1.** *Let  $G$  be a perfect group having a cyclic subgroup  $C_2$  of order 2,  $Y$  the complex projective space associated with the complex  $G$ -module*

$$V = \mathbb{C}^{\oplus m+1} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n},$$

*where  $m \geq 0$  and  $n \geq 1$ , and  $U$  the  $G$ -tubular neighborhood of  $Y^G$ .*

- (1)  *$Y$  satisfies the gap condition for  $(\{e\}, C_2)$  if and only if  $m + 1 = n$ .*
- (2)  *$U$  satisfies the gap condition for  $(\{e\}, C_2)$  if and only if  $m + 1 \leq n$ .*
- (3) *If  $m + 1 \leq n$  then  $U$  satisfies the strong gap condition for  $(H, K)$  such that  $\{e\} \neq H < K \leq G$  and  $|K : H| \geq 3$ .*

*Proof.* We readily see that  $Y^G = P_{\mathbb{C}}(\mathbb{C}^{m+1}) = P_{\mathbb{C}}^m$  and  $Y^{C_2}$  has two connected components

$$Y_a^{C_2} = P_{\mathbb{C}}(\mathbb{C}^{m+1} \oplus ((\mathbb{C}[G] - \mathbb{C})^{C_2})^{\oplus n}) = P_{\mathbb{C}}^{m+n(|G|/2-1)} \quad \text{and}$$

$$Y_b^{C_2} = P_{\mathbb{C}}(((\mathbb{C}[G] - \mathbb{C})_{C_2})^{\oplus n}) = P_{\mathbb{C}}^{n|G|/2-1}.$$

Thus we have  $\dim Y = 2m - 2n + 2n|G|$ ,  $\dim Y_a^{C_2} = 2m - 2n + n|G|$  and  $\dim Y_b^{C_2} = n|G| - 2$ . Note the equivalences

- $2(2m - 2n + n|G|) < 2m - 2n + 2n|G| \iff m < n$
- $2(n|G| - 2) < 2m - 2n + 2n|G| \iff n - 2 < m$ .

Thus  $Y$  satisfies the gap condition for  $(\{e\}, C_2)$  if and only if  $n - 2 < m < n$ , namely  $m + 1 = n$ . Since  $\dim U^{C_2} = \dim Y_a^{C_2}$ ,  $U$  satisfies the gap condition for  $(\{e\}, C_2)$  if and only if  $m < n$ , namely  $m + 1 \leq n$ .

For any  $H \leq G$ ,  $U^H$  is connected. Let  $y$  denote the point  $[1, 0, \dots, 0]$  in  $Y = P_{\mathbb{C}}(\mathbb{C} \oplus \mathbb{C}^{\oplus m} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n})$ . The tangential representation  $T_y(Y)$  is isomorphic to  $\mathbb{C}^{\oplus m} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n}$  as complex  $G$ -modules. Since  $\dim U^H = \dim T_y(Y)^H$ , we get

$$\dim U^H = 2\{m + n(|G|/|H| - 1)\} = 2m - 2n + 2n|G|/|H|$$

and

$$\dim U^H - 2(\dim U^K + 1) = 2\{n - (m + 1)\} + 2n|G|\{|K| - 2|H|\}/(|H||K|).$$

In the case  $n \geq m + 1$  and  $|K| \geq 3|H|$ , we conclude  $2(\dim U^K + 1) < \dim U^H$ .  $\square$

**Theorem 2.2** (Disk Theorem). *Let  $G$  be a nontrivial perfect group,  $F \in \mathfrak{F}(\mathfrak{D})$ ,  $F_0$  a connected component of  $F$  with  $x_0 \in F_0$ ,  $m = \dim F_0$ , and  $W$  a real  $G$ -module*

with  $\dim W^G = m$ . Then there exists a smooth  $G$ -action on the disk  $D$  of dimension  $\dim W + N(|G| - 1)$  for some integer  $N \geq 0$  satisfying the following conditions.

- (1)  $D^G = F$ .
- (2)  $T_{x_0}(D)$  is isomorphic to  $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus N}$  as real  $G$ -modules.
- (3)  $D$  satisfies the strong gap condition for arbitrary pair  $(H, K)$  such that  $H < K \leq G$ .

**Corollary 2.3.** Let  $G, F \in \mathfrak{F}(\mathfrak{D})$ ,  $F_0, x_0 \in F_0$ ,  $m = \dim F_0$ ,  $W, D$  and  $N$  be as in Theorem 2.2. Let  $n$  be an arbitrary integer  $\geq N$ . Then there exists a smooth  $G$ -action on the disk  $S$  of dimension  $\dim W + n(|G| - 1)$  satisfying the following conditions.

- (1)  $S^G = F \amalg F'$  and  $F'$  is diffeomorphic to  $F$ .
- (2)  $T_{x_0}(S)$  is isomorphic to  $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$  as real  $G$ -modules.
- (3)  $S$  satisfies the strong gap condition for arbitrary pair  $(H, K)$  such that  $H < K \leq G$ .

*Proof.* We set  $X = D \times D((\mathbb{R}[G] - \mathbb{R})^{\oplus(n-N)})$ . Let  $S$  be the double of  $X$ . Then  $S$  satisfies the desired conditions.  $\square$

### 3. DELETING THEOREM AND REALIZATION THEOREM

In this section we will give a deleting theorem and a realization theorem. The latter is obtained from the former and Corollary 2.3. Our main result Theorem 1.1 follows from the realization theorem.

Let  $A_5$  denote the alternating group on 5 letters 1, 2, 3, 4, 5, and let  $A_4$  denote the alternating group on 4 letters 1, 2, 3, 4. Unless otherwise stated, we use the notation:

- $C_2 = \langle (1, 2)(3, 4) \rangle$
- $D_4 = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$
- $C_3 = \langle (1, 2, 5) \rangle$
- $D_6 = \langle (1, 2, 5), (1, 2)(3, 4) \rangle$
- $C_5 = \langle (1, 3, 4, 2, 5) \rangle$
- $D_{10} = \langle (1, 3, 4, 2, 5), (1, 2)(3, 4) \rangle$ .

These groups and  $A_4$  are regarded as subgroups of  $A_5$ .

Throughout this section, let  $G$  be  $A_5$ . Then  $N_G(C_2) = D_4$ ,  $N_G(D_4) = A_4$ ,  $N_G(C_3) = D_6$ ,  $N_G(C_5) = D_{10}$ , and any maximal proper subgroup of  $G$  is conjugate to one of  $A_4$ ,  $D_{10}$ ,  $D_6$ . We can readily show

**Proposition 3.1.** *Let  $H$  be a maximal proper subgroup of  $G$ . Then any two subgroups of  $H$  are conjugate in  $G$  if and only if they are conjugate in  $H$ .*

**Proposition 3.2.** *Let  $\alpha$  be the element of the Burnside ring  $\Omega(G)$  given by*

$$\alpha = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/\{e\}].$$

*Then for any proper subgroup  $H < G$ ,  $\text{res}_H^G \alpha$  coincides with  $[H/H]$  in  $\Omega(H)$ .*

**Theorem 3.3** (Deleting Theorem). *(Let  $G = A_5$ .) Let  $Y$  be a compact connected smooth  $G$ -manifold of dimension  $\geq 5$ , with  $|\pi_1(Y)| < \infty$ , and with a decomposition  $Y^G = Y_0^G \amalg Y_1^G$  such that  $\partial Y_0^G = \emptyset$ . Let  $U$  be the  $G$ -tubular neighborhood of  $Y_0^G$ . Suppose  $U$  satisfies the gap condition for  $(\{e\}, C_2)$ ,  $(\{e\}, C_3)$  and  $(\{e\}, C_5)$ , and the cobordism gap condition for  $(C_2, D_6)$ ,  $(C_2, D_{10})$  and  $(C_3, A_4)$ . Then there exists a smooth  $G$ -manifold  $X$  possessing the following properties.*

- (1)  $X^G = Y_1^G$ .
- (2)  $X$  is homotopy equivalent to  $Y$ .
- (3)  $X^H$  is diffeomorphic to  $Y^H$  for any  $H$  such that  $\{e\} \neq H < G$ .
- (4) In the case that  $\dim Y \equiv 0 \pmod{2}$  and  $\pi_1(Y) = 1$ ,  $X$  is diffeomorphic to  $Y$ .

*Guideline for Proof.* There exists a compact connected smooth  $G$ -submanifold  $U_1$  of  $Y \setminus \partial Y$  with  $U \subset U_1$  such that  $G$  freely acts on  $U_1 \setminus U$  and the inclusion induced homomorphism  $\pi_1(U_1) \rightarrow \pi_1(Y)$  is an isomorphism. First, we construct a  $G$ -framed map  $f_1 : X_1 \rightarrow Y$  rel.  $Y \setminus \overset{\circ}{U}_1$  (i.e.,  $X_1 \supset Y \setminus \overset{\circ}{U}_1$ ,

$$f_1|_{Y \setminus \overset{\circ}{U}_1} : Y \setminus \overset{\circ}{U}_1 \rightarrow Y \setminus \overset{\circ}{U}_1$$

is the identity map, and  $f_1(X_2) \subset U_1$ , where

$$X_2 = X_1 \setminus (Y \setminus \overset{\circ}{U}_1)^\circ \setminus \partial Y.$$

Next we convert  $f_2 = f_1|_{X_2} : X_2 \rightarrow U_1$ , to a  $G$ -framed map  $f_3 : X_3 \rightarrow U_1$  such that  $f_3$  is a homotopy equivalence by  $G$ -surgeries rel.  $\partial U_1$  of isotropy types  $(H)$

for  $H < G$ . The construction of a  $G$ -framed map is discussed in Section 4. We perform the  $G$ -surgeries of isotropy types  $(H)$  with  $\{e\} < H < G$  by means of the reflection method in [8]. We do the  $G$ -surgeries of isotropy type  $(\{e\})$  by showing triviality of the algebraic  $G$ -surgery obstruction in the relevant Bak group described in [9] with the  $G$ -cobordism invariance property given in [10] and the induction-restriction property presented in [13, 4].  $\square$

**Theorem 3.4** (Realization Theorem). *(Let  $G = A_5$ .) Let  $W$  a real  $G$ -module with  $\dim W^G = m$ ,  $N$  an integer possessing the property described in Theorem 2.2, and  $Z$  a compact connected smooth  $G$ -manifold of dimension  $\geq 5$  such that  $\partial Z^G = \emptyset$ ,  $V$  the  $G$ -tubular neighborhood of  $Z^G$ ,  $z_0$  a point in  $Z^G$ , and  $n$  an integer  $\geq N$ . Suppose  $|\pi_1(Z)| < \infty$  and  $T_{z_0}(Z)$  is isomorphic to  $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$ . Further suppose  $V$  satisfies the gap condition for  $(\{e\}, C_2)$ ,  $(\{e\}, C_3)$  and  $(\{e\}, C_5)$ , and the cobordism gap condition for  $(C_2, D_6)$ ,  $(C_2, D_{10})$ , and  $(C_3, A_4)$ . Let  $F, F_0, x_0 \in F_0$  be as in Theorem 2.2. Then there exists a compact  $G$ -manifold  $X$  satisfying the following conditions.*

- (1)  $X^G$  is diffeomorphic to  $F$ .
- (2)  $X$  is homotopy equivalent to  $Z$ .
- (3) In the case that  $\dim Z \equiv 0 \pmod{2}$  and  $\pi_1(Z) = 0$ ,  $X$  is diffeomorphic to  $Z$ .

*Proof.* Take the smooth  $G$ -action on the sphere  $S$  described in Corollary 2.3 with  $x_0 \in F_0 \subset F$  and  $S^G = F \amalg F'$  such that  $T_{x_0}(S) \cong W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$  and  $F \cong F'$ . Let  $Y$  be the connected sum of  $S$  and  $Z$  at points  $x_0$  and  $z_0$ . By setting  $Y_0^G = F \# Z^G$  and  $Y_1^G = F'$  we have  $Y^G = Y_0^G \amalg Y_1^G$ . Note  $Y_0^G$  is without boundary. The tubular neighborhood  $U$  of  $Y_0^G$  satisfies the gap condition for  $(\{e\}, C_2)$ ,  $(\{e\}, C_3)$  and  $(\{e\}, C_5)$ , and the cobordism gap condition for  $(C_2, D_6)$ ,  $(C_2, D_{10})$ , and  $(C_3, A_4)$ . Deleting the fixed point submanifold  $F_0^G$  from  $Y$  by means of Theorem 3.3, there exists a  $G$ -manifold  $X$  satisfying the desired conditions in the theorem, where  $X^G = F' \cong F$ .  $\square$

#### 4. EQUIVARIANT COHOMOLOGY THEORY $\omega(\bullet)_G^*$ AND $G$ -FRAMED MAPS

Equivariant surgeries are operated on smooth  $G$ -manifolds, but more precisely on  $G$ -framed maps. T. Petrie gave an idea to construct  $G$ -framed maps by using



the equivariant cohomology theory  $\omega_G^*(\bullet)$  and the Burnside ring  $\Omega(G)$ . In order to construct our  $G$ -framed maps, we employ a modification given in [11] of Petrie's construction.

Let  $\Omega(G)$  denote the Burnside ring, i.e.

$$\Omega(G) = \{[X] \mid X \text{ is a finite } G\text{-CW complex}\},$$

where  $[X] = [Y]$  if and only if  $\chi(X^H) = \chi(Y^H)$  for all  $H \leq G$ . The next proposition is well known, see [2, 20, 1].

**Proposition 4.1.** *Let  $G$  be a nontrivial perfect group. Then there exists an idempotent  $\beta$  in the Burnside ring  $\Omega(G)$  such that  $\chi_G(\beta) = 1$  and  $\chi_H(\beta) = 0$  for all  $H \neq G$ .*

Let  $\mathcal{S}(G)$  and  $\mathcal{S}(G)_{\max}$  denote the set of all subgroups and all maximal proper subgroups of  $G$ , respectively. For a subgroup  $H$  of  $G$ ,  $(H)$  stands for the  $G$ -conjugacy class containing  $H$ , i.e.

$$(H) = \{gHg^{-1} \mid g \in G\}.$$

The  $\beta$  in Proposition 4.1 has the form

$$(4.1) \quad \beta = [G/G] - \sum_{(K) \subset \mathcal{S}(G)_{\max}} [G/K] - \sum_{(H) \subset \mathcal{F}} a_H [G/H],$$

for some  $G$ -invariant lower closed  $\mathcal{F} \subset \mathcal{S}(G) \setminus (\mathcal{S}(G)_{\max} \cup (G))$  and  $a_H \in \mathbb{Z}$ .

**Proposition 4.2.** *Let  $G = A_5$  and let  $\alpha$  be the element of the Burnside ring  $\Omega(G)$  given by*

$$\alpha = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/\{e\}].$$

*Then for any proper subgroup  $H < G$ ,  $\text{res}_H^G \alpha$  coincides with  $[H/H]$  in  $\Omega(H)$ .*

**Proposition 4.3.** *Let  $G$  be  $A_5$ . Then there exists a finite  $G$ -CW complex  $Z$  fulfilling the following conditions.*

- (1)  $Z^G = \{z_0, z_1\}$ .
- (2)  $\bigcup_{H \in \mathcal{S}(G)_{\max}} Z^H = \{z_0, z_1\} * \mathcal{S}(G)_{\max}$ , where each subgroup in  $\mathcal{S}(G)_{\max}$  is regarded as a point. Thus  $Z^H$  is homeomorphic to the 1-dimensional disk  $[-1, 1]$  for each  $H \in \mathcal{S}(G)_{\max}$ .

$$(3) \ Z^{D_4} = Z^{A_4} \text{ and } Z^{C_5} = Z^{D_{10}}.$$

$$(4) \ Z^H \text{ is contractible for any subgroup } H < G.$$

The  $G$ -CW complex  $Z$  in this lemma is constructed using Oliver-Petrie's  $G$ -CW-surgery theory [21] and the wedge sum technique [16].

Let  $M_n = \mathbb{C}[G]^{\oplus n}$  and let  $M_n^\bullet$  be the one-point compactification of  $M_n$ , hence we write  $M_n^\bullet = M_n \cup \{\infty\}$ . For a finite  $G$ -CW complex  $X$  with base point in  $X^G$ ,

$$\bar{\omega}_G^0(X) = \lim_{n \rightarrow \infty} [X \wedge M_n^\bullet, M_n^\bullet]_0^G,$$

where  $[-, -]_0^G$  stands for the set of all homotopy classes of maps in the category of pointed  $G$ -spaces. For a finite  $G$ -CW complex  $Z$ ,  $\omega_G^0(Z)$  is defined to be  $\bar{\omega}_G^0(Z^+)$ , where

$$Z^+ = Z \amalg (G/G)$$

and  $G/G$  is regarded as the base point of  $Z^+$ .

For the set  $S$  of all powers  $\beta^k$ ,  $k \in \mathbb{N}$ , the restriction  $j^* : S^{-1}\omega_G^0(Z) \rightarrow S^{-1}\omega_G^0(Z^G)$  induced by the inclusion map  $j : Z^G \rightarrow Z$  is an isomorphism. It is obvious that for any element  $\gamma \in \omega_H^0(Z)$  and any proper subgroup  $H$  of  $G$ ,  $\text{res}_H^G \beta \cdot \gamma = 0_Z$  in  $\omega_H^0(Z)$ .

**Lemma 4.4.** *Let  $G$  be a nontrivial perfect group,  $\beta$  the element in Proposition 4.1, and  $Z$  a finite  $G$ -CW complex with*

$$(4.2) \quad Z^G = \{z_0, z_1\}.$$

*Then there exists an element  $\gamma \in \omega_G^0(Z)$  such that  $\gamma|_{z_0} = \beta$  and  $\gamma|_{z_1} = 0_{z_1}$  in  $\Omega(G)$  and  $\beta\gamma = \gamma$ . In addition, for any proper subgroup  $H < G$ , there exists a 'homotopy'  $\Gamma_H \in \omega_H^0(Z \times I)$  from  $\text{res}_H^G \gamma$  to  $0_Z$ , rel.  $z_1 \times I$ , i.e.  $\Gamma_H|_{Z \times \{0\}} = \text{res}_H^G \gamma$ ,  $\Gamma_H|_{Z \times \{1\}} = 0_Z$ , and  $\Gamma_H|_{z_1 \times I} = 0_{z_1 \times I}$ , such that  $\text{res}_H^G \beta \cdot \Gamma_H = \Gamma_H$ . Moreover, for any pair of distinct proper subgroups  $H$  and  $K$  of  $G$ , there exists a 'homotopy'  $\bar{\Gamma}_{H,K} \in \omega_{H \cap K}^0(Z \times I \times I)$  from  $\text{res}_{H \cap K}^H \Gamma_H$  to  $\text{res}_{H \cap K}^K \Gamma_K$ , rel.  $z_1 \times I \times I$  and  $Z \times \partial I \times I$ .*

As a next step, consider the elements  $1_Z - \gamma \in \omega_G^0(Z)$ ,  $1_{Z \times I} - \Gamma_H \in \omega_H^0(Z \times I)$ , and  $1_{Z \times I \times I} - \bar{\Gamma}_{H,K} \in \omega_{H \cap K}^0(Z \times I \times I)$ . Recall that an element  $\alpha \in \omega_G^0(Z)$  is represented by a  $G$ -map

$$Z^+ \wedge M^\bullet \rightarrow M^\bullet$$

preserving the base point  $\infty$ , where  $M = \mathbb{C}[G]^{\oplus n}$  for some  $n$ .

**Lemma 4.5.** *Let  $G$  be a nontrivial perfect group,  $\beta$  the element in Proposition 4.1, and  $Z$  a finite  $G$ -CW complex with  $Z^G = \{z_0, z_1\}$ . Then there exist maps  $\alpha$ ,  $A_H$ , and  $\bar{A}_{H,K}$  satisfying the following conditions (1)–(3), where  $H$  and  $K$  range all proper subgroups of  $G$  such that  $H \neq K$ .*

- (1)  $\alpha$  is a map of pointed  $G$ -spaces  $Z^+ \times M^\bullet \rightarrow M^\bullet$  such that  $[\alpha] = 1 - \gamma$  for the  $\gamma$  above, and  $\alpha|_{\{z_1\}^+ \wedge M^\bullet} = id_{\{z_1\}^+ \wedge M^\bullet}$ .
- (2)  $A_H$  is a homotopy of pointed  $H$ -spaces  $(Z \times I)^+ \wedge M^\bullet \rightarrow M^\bullet$  from  $\alpha$  to  $1_Z$ , where  $1_Z : Z^+ \wedge M^\bullet \rightarrow M^\bullet$  and  $1_Z(z, v) = v$  for all  $z \in Z$  and  $v \in M$ , rel.  $\{z_1\}^+ \wedge M^\bullet$ .
- (3)  $\bar{A}_{H,K}$  is a homotopy of pointed  $H \cap K$ -spaces  $((Z \times I) \times I)^+ \wedge M^\bullet \rightarrow M^\bullet$  from  $A_H$  to  $A_K$  rel.  $(z_1 \times I)^+ \wedge M^\bullet$  and  $(Z \times \partial)^+ \wedge M^\bullet$ .

**Proposition 4.6.** *Let  $G = A_5$  and  $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$ . Let  $Z$  be the finite  $G$ -CW complex in Proposition 4.3. Then there exist maps  $\alpha$ ,  $A_H$ , and  $\bar{A}_{H,K}$  of Lemma 4.5 satisfying the additional conditions:*

- (1)  $\alpha|_{z_0}^{-1}(0)^G = \emptyset$  and  $|(\alpha|_{z_0}^{-1}(0))^H| = 1$  for  $H \in \mathcal{K}$ .
- (2) For each  $H \in \mathcal{K}$ , there exists a connected component  $X(H)$  of  $\alpha^{-1}(0)$  containing both  $(\alpha|_{z_0}^{-1}(0))^H$  and  $(z_1, 0)$  such that  $\alpha$  is transversal on  $X(H)$  to  $0 \subset M$ , the normal derivative of  $\alpha$  on  $X(H)$  is the identity, and the projection  $Z \times M \rightarrow Z$  diffeomorphically maps  $X(H)$  to  $Z^H$ .
- (3) For each pair of  $H \in \mathcal{S}(G)_{\max}$  and  $L \in \mathcal{K}$  with  $L \leq H$ , there exists a connected component  $W(H, L)$  of  $A_H^{-1}(0)$  containing  $X(H)^L \times \{0\}$  ( $\subset Z \times M \times I$ ) and  $Z^L \times 0 \times \{1\}$  ( $\subset Z \times M \times I$ ) such that  $A_H$  is transversal on  $W(H, L)$  to  $0 \subset M$ , the normal derivative of  $A_H$  on  $W(H, L)$  is the identity, and the projection  $Z \times I \times M \rightarrow Z \times I$  diffeomorphically maps  $W(H, L)$  to  $Z^L \times I$ .

A  $G$ -framed map  $\mathbf{f} = (f, b)$  consists of a  $G$ -map  $f : X \rightarrow Y$  such that  $X$  and  $Y$  are compact smooth  $G$ -manifold and  $f(\partial X) \subset \partial Y$ , and an isomorphism  $b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \rightarrow f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^m)$  of real  $G$ -vector bundles for some integer  $m \geq 0$ . In the following we suppose  $Y$  is connected and  $f : (X, \partial X) \rightarrow (Y, \partial Y)$  is of degree 1.

**Lemma 4.7.** *Let  $Y$  be a compact smooth  $G$ -manifold with a decomposition  $Y^G = Y_0^G \amalg Y_1^G$  such that  $\partial Y_0^G = \emptyset$ . Let  $U$  be the  $G$ -tubular neighborhood of  $Y_0^G$ . Then there exist a  $G$ -framed map  $\mathbf{f} = (f, b)$ ,  $H$ -framed cobordisms  $\mathbf{F}_H = (F_H, B_H) : \mathbf{f} \sim \mathbf{id}_Y$ , rel.  $Y \setminus \overset{\circ}{U}$  for  $H < G$ , and  $H \cap K$ -framed cobordisms  $\overline{\mathbf{F}}_{H,K} = (\overline{F}_{H,K}, \overline{B}_{H,K}) : \mathbf{F}_H \sim \mathbf{F}_K$  rel.  $((Y \setminus \overset{\circ}{U}) \times I) \cup (Y \times \partial I)$ , for  $H, K < G$  such that  $H \neq K$ , where*

$$f : X \rightarrow Y,$$

$$b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \rightarrow f^*(Y) \oplus \varepsilon_X(\mathbb{R}^m),$$

$$F_H : W_H \rightarrow Y \times I,$$

$$B_H : T(W_H) \oplus \varepsilon_{W_H}(\mathbb{R}^m) \rightarrow F_H^*T(Y \times I) \oplus \varepsilon_{W_H}(\mathbb{R}^m),$$

$$\overline{F}_{H,K} : \overline{W}_{H,K} \rightarrow Y \times I \times I,$$

$$\overline{B}_{H,K} : T(\overline{W}_{H,K}) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m) \rightarrow \overline{F}_{H,K}^*T(Y \times I \times I) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m),$$

for some integer  $m > 0$ .

This lemma is obtained by the arguments in [11].

**Lemma 4.8.** *Let  $G = A_5$  and  $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$ . Then the framed maps  $\mathbf{f}$ ,  $\mathbf{F}_H$  and  $\overline{\mathbf{F}}_{H,K}$  in Lemma 4.7 can be chosen so that  $X^L$  and  $W_H^L$  are  $N_H(L)$ -diffeomorphic to  $Y^L$  and  $Y^L \times I$ , respectively, for all  $H, K \in \mathcal{S}(G)_{\max}$  and all  $L \in \mathcal{K}$  with  $L \leq H$ .*

This modification is achieved by using Proposition 4.6 and the reflection method in [8].

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